SYSTEMATIC DERIVATION OF VARIATIONAL PRINCIPLES IN ELECTROMAGNETIC FIELD THEORY

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Abstract

Variational principles for a wide class of functionals can be constructed in a routine fashion. The procedure is illustrated by deriving a stationary expression for the resonant frequencies of a cavity, Shwinger's variational principle, and a variational principle for a function itself.

Introduction

Different procedures - such as Rumsey's reaction concept[1], Cairo and Kahn's transpose-operator and field technique[2], and Hamilton's principle of least action[3] - have been employed in the literature to derive variational principles in electromagnetic field problems. Recent work on variational principles in mathematical physics[4] should enable one to construct in a novel way stationary expressions for a wide class of functionals $P(\vec{E})$, where $\vec{E}$ is an unknown (vector) function. This is accomplished by accounting for each of the equations (constraints) that define $\vec{E}$ via a Lagrange undetermined multiplier, which can be a constant λ, a function $\vec{F}(x,y,z)$, and an operator $\overleftrightarrow{T}$. For example if $\vec{E}$ represents a normalized electric field, which obeys the wave equation, then the normalization constraint will require a constant multiplier while the wave equation, which is a constraint at each point in space, will require an undetermined function.

The method of constraints has the advantage that it is systematic and applicable to a wide class of problems. The following three examples will illustrate the details of the procedure.

Resonant Frequencies of Cavity

Here we derive Berk's[5] stationary expression for the resonant frequencies of a cavity in terms of the electric field $\vec{E}$. Now,

$$-\nabla(\tfrac{1}{\mu}\nabla\times\vec{E})+\varepsilon(\omega/c)^2\vec{E}=[-H_0+\varepsilon(\omega/c)^2]\vec{E}=0, \qquad (1)$$

where

$$H_0 \equiv \nabla\times\frac{1}{\mu}\nabla\times \quad . \qquad (2)$$

μ,ε,ω,c are the relative permeability, permittivity, angular frequency, and speed of light, respectively. The problem has two constraints, Eq. (1), and the normalization of $\vec{E}$, $\int\vec{E}^{\dagger}\vec{E}d\tau = 1$. With the dagger representing the Hermitian adjoint and $d\tau$ the element of volume. The variational principle can now be written down routinely

$$[\int\vec{E}^{\dagger}\varepsilon(\omega/c)^2\vec{E}d\tau]\text{var} \equiv [\int\vec{E}^{\dagger}H_0\vec{E}d\tau]\text{var} = \int\vec{E}_t^{\dagger}H_0\vec{E}_t d\tau$$
$$+\lambda_t\int[\vec{E}_t^{\dagger}\vec{E}_t-1]d\tau+\int\vec{F}_t^{\dagger}[H_0-\varepsilon\omega^2/c^2]\vec{E}_t d\tau$$
$$+\int\vec{F}_t[(H_0-\varepsilon\omega^2/c^2)\vec{E}_t]^{\dagger}d\tau, \qquad (3)$$

where

$$\vec{E}_t = \vec{E}+\delta\vec{E},\ \vec{F}_t = \vec{F}+\delta\vec{F}, \qquad \lambda_t = \lambda+\delta\lambda \qquad (4)$$

are the trial electric field, constraint function and constraint multiplier, respectively. λ and $\vec{F}$ are determined by the requirement that

$$\delta\int\vec{E}^{\dagger}H_0\vec{E}d\tau = \int\vec{E}^{\dagger}H_0\vec{E}d\tau-\int\vec{E}_t^{\dagger}H_0\vec{E}_t d\tau$$

vanishes to first order in $\delta\vec{E}$. Neglecting second-order terms, it follows for μ,ε Hermitean tensors that

$$[H_0-\varepsilon(\omega/c)^2]\vec{F} + H_0\vec{E}+\lambda\vec{E} = 0 \qquad (5)$$

Multiplying Eq. (5) with $\vec{E}$ and integrating, one obtains

$$\lambda = -\int\vec{E}^{\dagger}H_0\vec{E}d\tau \qquad (6)$$

Therefore

$$[H_0-\varepsilon(\omega/c)^2]\vec{F} = \vec{E}\int\vec{E}\cdot H_0\vec{E}d\tau-H_0\vec{E} \qquad (7)$$

With $\int\vec{E}_t^{\dagger}\vec{E}_t d\tau= 1$ and the help of (7) [replacing $\vec{F},\vec{E}$ by $\vec{F}_t,\vec{E}_t$], Eq. (3) reduces to

$$[\int\vec{E}^{\dagger}\varepsilon(\omega/c)^2\vec{E}]\text{var} = \int\vec{E}_t^{\dagger}H_0\vec{E}_t d\tau \qquad (8)$$

Thus

$$(\omega/c)^2 = \int\vec{E}^{\dagger}H_0\vec{E}d\tau/[\int\vec{E}^{\dagger}\varepsilon\vec{E}d\tau] \qquad (9)$$

is stationary for $\vec{E}_{tan}=0$ on conducting surfaces, the above equation reduces to Berk's expression

$$(\omega/c)^2 = \int(\nabla\times\vec{E})^{\dagger}\mu^{-1}(\nabla\times\vec{E})d\tau/[\int\vec{E}^{\dagger}\varepsilon\vec{E}d\tau] \qquad (10)$$

Schwinger's Variational Principle

As another illustration of the constraint method, we show how to construct Schwinger's variational principle for the scattering of an electromagnetic wave by a dielectric obstacle in a waveguide[6].

For simplicily, the obstacle is symmetric with respect to a plane perpendicular to the direction of propagation z. The problem can be analyzed in terms of even and odd waves. We will consider only the odd case. The wave equation can be written

$$-\nabla\times\nabla\times\vec{E}+\varepsilon(\omega/c)^2\vec{E} = [-H+(\omega/c)^2]\vec{E} =0, \qquad (11)$$

where

$$H = \nabla\times\nabla+(1-\varepsilon)(\omega/c)^2 \quad ; \qquad (12)$$

we assumed $\mu=1$. The odd standing wave solution of $\vec{E}$ satisfies $\vec{E}_0(0)=0$ and has the asymptotic form for $z\to\infty$

$$\vec{E}_0(x,y,z) = \vec{e}(x,y)[\sin(kz+\theta)-\tan(\eta_0-\theta)\cos(kz+\theta)], \qquad (13)$$

where $\vec{e}(x,y)$ is the form function of the propagating mode, k is the propagation constant, and[7]

$$\tan(\eta_0-\theta) = k^{-1}\int(\varepsilon-1)(\omega/c)^2\vec{E}\cdot\vec{e}\sin(kz+\theta), \qquad (14)$$

$$\tan\eta_0 = -i(Z_{11}-Z_{12}) \quad . \qquad (15)$$

θ is a parameter which introduces an additional degree of freedom and will be utilized later on. η_0 is the (so-called) odd phase shift, and Z_{11} and Z_{12} are elements of the impedance network of the obstacle. Now from Eqs (11) and (14),

$$\tan(\eta_0-\theta)=k^{-1}\int\vec{e}\sin(kz+\theta)\cdot[H_0-(\omega/c)^2]\vec{E}_0 d\tau, \quad (16)$$

where

$$H_0 = H+W \quad (17a)$$

$$W = (\varepsilon-1)(\omega/c)^2 \quad (17b)$$

The variational principle for tan $(\eta_0-\theta)$ reads

$$k[\tan(\eta_0-\theta)]\,\text{var} = \int\vec{e}\sin(kz+\theta)\cdot[H_0-(\omega/c)^2]\vec{E}_{0t}d\tau - \int\vec{F}_t\cdot[H-(\omega/c)^2]\vec{E}_{0t}d\tau \quad (18)$$

$\vec{F}$ is specified by letting the first variation of (18) be zero. One obtains

$$\int\vec{e}\sin(kz+\theta)\cdot[H_0-(\omega/c)^2]\delta\vec{E}_{0t}d\tau - \int\vec{F}\cdot[H-(\omega/c)^2]\delta\vec{E}_{0t}d\tau = 0. \quad (19)$$

Integrating by parts it follows that

$$[H-(\omega/c)^2]\vec{F} = 0, \quad (20)$$

and that

$$\vec{F} = \vec{E} \quad (21)$$

Therefore

$$k[\tan(\eta_0-\theta)]\,\text{var.}=k\tan(\eta_{0t}-\theta)-\int\vec{E}_{0t}\cdot[H-(\omega/c)^2]\vec{E}_{0t}d\tau \quad (22)$$

For $\theta=0$, this is equivalent to Kohn's variational principle in quantum mechanics[8].

Letting $\theta=\pi/2$ and adding the <u>second-order</u> term $\int[(H-\omega^2/c^2)\vec{E}_{0t}]^2W^{-1}d\tau$ to (22), one obtains

$$(k\cot\eta_0)\text{var} = k\cot\eta_{0t}+\int\vec{E}_{0t}\cdot[H-(\omega/c)^2]\vec{E}_{0t}d\tau + \int[(H-\omega^2/c^2)\vec{E}_{0t}]^2W^{-1}d\tau \quad (23)$$

As shown by Kato[8], this is equivalent to Schwinger's variational principle, p. 51 Ref. [6]. Namely,

$$\cot\eta_0 = \frac{1}{Z_{11}-Z_{12}}\,\frac{i}{k} = \frac{\int W\vec{E}_0{}^2d\tau-\int W(\vec{r})\vec{E}_0(\vec{r})\cdot\overleftrightarrow{G}_0W(\vec{r}\,')\vec{E}_0(\vec{r}\,')d\tau d\tau'}{[\int W\vec{e}\sin kz\cdot\vec{E}_0d\tau]^2} \quad (24)$$

where the Green's function $\overleftrightarrow{G}_0(x,y,z;x,y,z)$ satisfies the equation

$$[-\nabla\times\nabla\times + (\omega/c)^2]\overleftrightarrow{G}_0 = -\overleftrightarrow{I}\delta(\vec{r}-\vec{r}\,') \quad (25)$$

and the appropriate boundary conditions. $\overleftrightarrow{I}$ is the idemfactor.

Variational Function

The construction of a variational function $\vec{E}_{var}$ with no first-order error $\delta\vec{E}$ in $\vec{E}_t$ will serve as a final example. The functional $P(\vec{E}_{var})$, evaluated with $\vec{E}_{var}$, will clearly be a variational expression of the functional $P(\vec{E})$. Proceeding as before, one writes

$$\vec{E}_{var}(\vec{r})= \vec{E}_t +\int\overleftrightarrow{\Gamma}(\vec{r},\vec{r}\,')[-H(\vec{r})+(\omega/c)^2]\vec{E}_t(\vec{r}\,')d\tau' \quad (26)$$

The requirement that (26) contain no first-order errors $\delta\vec{E}$ and $\delta\overleftrightarrow{\Gamma}$ leads to

$$[-H+(\omega/c)^2]\overleftrightarrow{\Gamma} = -\overleftrightarrow{I}\delta(\vec{r}-\vec{r}\,'), \quad (27)$$

and certain boundary conditions on $\overleftrightarrow{\Gamma}$. In other words, $\overleftrightarrow{\Gamma}$ is a Green's function. The replacement of $\overleftrightarrow{\Gamma}$ in (26) by a trial Green's function $\overleftrightarrow{\Gamma}_t$ yields the desired $\vec{E}_{var}$.

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